Dark matter equation of state from rotational curves of galaxies

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We obtain a dark matter equation of state by using the rotation curves of galaxies only. Dark matter is modeled as a fluid component with pressure and we show that, within the Newtonian limit, the resulting equation of state is compatible with existing cosmological bounds. An upper bound on the central pressure of the dark matter is obtained, within the full general relativistic treatment, by demanding that the dark matter fulfills the positive energy condition.

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I. INTRODUCTION

Astrophysical observations of the movement of stars in galaxies, of the cosmic microwave background (CMB) anisotropies, the weak gravitational lensing of distant galaxies, among others, are all explained by the presence of dark matter, which has not shown any interaction with the baryonic matter except the gravitational one [1]. On the other hand, direct detection of dark matter demands it to have some interaction with the rest of particles of the standard model [2]. These two research lines seem to have contradictory points of view on the properties of the dark matter. In order to attempt to conciliate these seemingly opposite strategies, one can research on the possibility that the astrophysical description of dark matter could relax the pressure-less hypothesis, and allow some interaction with itself and with the baryonic matter.

For instance, at the cosmological level, Müller in [3] proposed a barotropic relation for the dark fluid, $p = \omega_{\rm DM}\,\rho$, instead of considering the usual dust case ($\omega_{\rm DM} = 0$). It was shown that in the case where the adiabatic sound speed coincides with the sound speed, the combined analysis of data from anisotropies of the CMB, supernovae Ia and matter power spectra strongly constrain $\omega_{\rm DM}$ to lay within the interval $[-1.5, 1.13] \times 10^{-6}$ (see also [4, 5]).

Since the strongest evidence for dark matter at galactic scales comes from the rotational curves, it is natural to look for a functional relation between pressure and energy density by modeling the dark matter halo as a perfect fluid. In the present work we show that indeed it is possible to derive (not to propose) such functional relation solely with the use of the rotational curve profiles. We performed a general relativistic treatment of the problem, as well as its Newtonian counterpart. The relativistic result gives a maximum value of the pressure at the center of the galaxy, assuming that dark matter should fulfill the positive energy condition. The general relativistic result contains, as expected, the Newtonian description as a particular case, and, in this limit, the derived equation of state is consistent with the bound obtained by Müller [3]. In fact, when the value of the pressure at the center is very small, there are no remarkable differences between both treatments, and the Newtonian model can be used with confidence. We found an analytical expression for the dark matter equation of state in the Newtonian limit as a function of the asymptotic circular velocity of the halo.

We have assumed spherical symmetry and a static and isotropic description for the dark matter fluid. These considerations reduce the number of unknowns, so that only one observation (the rotational curve profile) is enough to determine the equation of state. If the isotropic assumption is relaxed, then it is needed another observation to determine the equation of state. Such observation can be the gravitational lensing, see for instance [6–8].

The manuscript is organized as follows: In section II we present the general relativity equations for a perfect fluid and discuss how one of the metric coefficients is completely determined by the rotational curve profile; we also present the Newtonian treatment. In section III we compute the equation of state by means of the rotational curves of LSB galaxies fitted by the Salucci profile [9]. We finally discuss some of the implications of the derived equation of state and conclude in section IV.

II. PERFECT FLUID HALO

We consider a static and spherically symmetric spacetime in General Relativity, described by the line element:

$$ds^2 = -e^{2\Phi/c^2}\, c^2\, dt^2 + \frac{dr^2}{1 - \frac{2\,G\,m}{c^2\,r}} + r^2\, d\Omega^2, \qquad (1)$$

where $d\Omega = d\theta^2 + \sin^2\theta d\varphi^2$. The gravitational potential $\Phi(r)$ and the mass function $m(r)$ are functions of the radial coordinate only.

One of the geometric potentials can be determined by the observations. Indeed, it can be obtained an expression relating the tangential velocities of test particles, v_t, following stable circular orbits in this spacetime, with

the gravitational potential Φ, by means of solving the geodesic equations for test particles in circular stable orbits (see [10–12] for details on the derivation). Such expression is

$$\frac{\Phi'}{c^2} = \frac{\beta^2}{r}, \tag{2}$$

where $\beta^2 = \frac{{v_t}^2}{c^2}$.

It is remarkable that the same expression is obtained by the Newtonian description, by simply equating the gravitational force with the centrifugal one for particles in circular orbit.

It is convenient to define dimensionless variables for the mass and the radial distance, namely $m = M_0\, n$ and $r = R_0\, x$, where M_0, R_0 are the characteristic scales for mass and distance of the system under study, and n is a dimensionless function related to the mass and x a dimensionless quantity related to the distance. With these scales, we also define a characteristic density, $\rho_0 = M_0/(4/3)\pi\, {R_0}^3$, and a characteristic pressure: $p_0 = \rho_0\, c^2$, so that $\rho = \rho_0\, \bar{\rho}$, and $p = p_0\, \bar{p}$, with $\bar{\rho}, \bar{p}$ dimensionless functions related to the density and the pressure respectively.

With these definitions, we rewrite the Einstein's equations and the conservation equation as follows:

$$\begin{aligned} n' &= 3\, x^2\, \bar{\rho} && (3) \\ \left(1 - q\, \frac{2\, n}{x}\right) \frac{\beta^2}{x} - q\, \frac{n}{x^2} &= 3\, q\, x\, \bar{p}, && (4) \\ \bar{p}' + (\bar{p} + \bar{\rho})\, \frac{\beta^2}{x} &= 0, && (5) \end{aligned}$$

where we have already used the relation (2) and prime stands for derivative with respect to x. Also, it was defined $q = M_0\, G/c^2\, R_0$ which is a dimensionless quantity.

Thus, to solve the system of a self-gravitating isotropic fluid, we are left with three unknowns: one geometric, the mass function, and two related to the dark fluid, the pressure and the energy density.

From observations, it is known that the terminal velocity is in general of the order of 100 Km/s, so that, using the Newtonian relation for the mass of the halo: $m(r) = r\, {v_t}^2/G$, and taking $G = 4.299\ 10^{-6}\, \frac{\mathrm{Kpc\,(km/s)^2}}{M_{\mathrm{Sun}}}$, and the characteristic distance in Kpc, we see that the characteristic mass of these systems is $10^{10} M_\odot$. Also, we find that the parameter q is of the order of 5×10^{-7}. We will use this value for q in our computations. With these values, the characteristic values of the density and pressure of the system are: $\rho_0 = 1.62\, 10^{-22}\, \frac{\mathrm{grm}}{\mathrm{cm}^3}$ and $p_0 = 1.46\, 10^{-1}\, \frac{\mathrm{grm}}{\mathrm{cm}\, s^2}$.

After some manipulation of equations (3, 5), we obtain an equation involving only the pressure, $\bar{p}$:

$$\bar{p}' + S(x)\, \bar{p} = T(x), \tag{6}$$

with

$$\begin{aligned} S(x) &= -\frac{2\, \beta^2\, \left(1 + \beta^2 - 2\, \beta^4 - x\, {\beta^2}'\right)}{(1+\beta^2)\, (1 + 2\, \beta^2)\, x}, \\ T(x) &= -\frac{\beta^2\, \left(\beta^2 + 2\, \beta^4 + x\, {\beta^2}'\right)}{3\, (1+\beta^2)\, (1 + 2\, \beta^2)\, x^3\, q}. \end{aligned} \tag{7}$$

The general solution for the pressure, expressed in terms of the observed rotational velocity profile, $\beta(x)$, has the form

$$\bar{p} = \frac{\int e^{\int^x S(x')\, dx'} T(x) dx + C}{e^{\int^x S(x')\, dx'}}, \tag{8}$$

where the value of the integration constant, C, is set by the appropriate boundary conditions.

The density and the mass of the dark perfect fluid can be directly computed from Eqs. (3-5), once the pressure is determined:

$$\bar{\rho} = \frac{\beta^2 + 2\, \beta^4 + x\, {\beta^2}'}{3\, (1 + 3\beta^2 + \beta^4)\, x^2\, q} - \frac{3 + 5\, \beta^2 - 2\, \beta^4 - 2\, x\, {\beta^2}'}{(1 + 3\beta^2 + \beta^4)}\, \bar{p}, \tag{9}$$

and

$$n = \frac{\beta^2}{(1 + 2\, \beta^2)\, q}\, x - 3\, \frac{x^3}{(1 + 2\, \beta^2)}\, \bar{p}. \tag{10}$$

Once we have presented the general relativistic treatment for the dark perfect fluid, it is interesting to have a look on the Newtonian description in order to asses its accuracy. Considering an static spherical dark matter distribution we have the Poisson's equation and the definition of the mass. To these equations, we add the Euler's equation in this case, for the co-movil observer. Thus, we obtain the following set of Newtonian equations:

$$\begin{aligned} {\nabla_x}^2\, \beta^2 &= 3\, q\, \bar{\rho}, && (11) \\ n &= \frac{\beta^2}{q}\, x && (12) \\ \bar{p}' &= -\bar{\rho}\, \frac{\beta^2}{x}, && (13) \end{aligned}$$

where the considerations for the defining dimensionless quantities are as those given above; we used the relation valid in the Newtonian limit, between the mass and the gravitational potential, $\Phi' = G\, m/r^2$, and we express this last one in terms of the observed velocity profile ratio, $\beta(x)$.

III. DARK MATTER EQUATION OF STATE FROM ROTATIONAL CURVES

The dark matter equation of state is directly computed from Eqs. (8) and (9) once the velocity profile $\beta^2(x)$ is given. We will consider an observational velocity profile

proposed by [9], which was obtained by adjusting 1023 galaxies. It is given as:

$$\beta^2 = {\beta_0}^2\,\frac{x^2}{x^2+a^2}\,, \tag{14}$$

where β_0, a are constants. β_0 stands for the ratio of the terminal velocity to the speed of light, and a determines how fast the velocity reaches a terminal value. Within the notation of [9], β_0 is a function of the luminosity of the galaxy given by ${\beta_0}^2 = \frac{V^2(R_{\rm opt})\left(1-\gamma(L)^2\right)\left(1+a^2\right)}{c^2}$, where $\gamma(L)^2 = 0.72+0.44\log L/L^*$. In our present work we will assume that both β_0^2 and a are simply constants that can be fitted to observational data.

It is worth to mention that one of the integrals for the pressure in Eq. (8), is solved analytically when the rotational curve profile Eq. (14) is used. Indeed:

$$e^{\int^x S(x')\,dx'} = \frac{\left(x^2+\beta_0^2x^2+a^2\right)^{\frac{2}{1+{\beta_0}^2}}}{(x^2+2\beta_0^2x^2+a^2)\,\left(x^2+a^2\right)^{(1-{\beta_0}^2)}}\,, \tag{15}$$

which can be inserted immediately in Eq. (8) and the equation of state can be computed. Indeed, once $\bar{p}(x)$ is computed, the density will be in this case given by

$$\bar{\rho}(x) = \frac{\beta_0^2(3a^2+(1+2\beta_0^2)x^2)}{3q(a^2+(1+\beta_0^2)x^2)(a^2+(1+2\beta_0^2)x^2)} - \frac{(3a^4+a^2(6+\beta_0^2)x^2+(3+5\beta_0^2-2\beta_0^4)x^4)}{a^4+a^2(2+\beta_0^2)x^2+(1+\beta_0^2+2\beta_0^4)x^4}\bar{p}(x)\,, \tag{16}$$

which is a decreasing function. If we demand that dark matter is regular matter that fulfills the positive energy condition, then it is sufficient to demand $\bar{\rho}(0) > 0$. Hence Eq. (16) implies an upper limit on the central pressure completely determined by the terminal velocity β_0 and a, which explicitly is:

$$\bar{p}(0) < \frac{\beta_0^2}{3a^2q}\,. \tag{17}$$

In order to advance with our method, we will use observational data for the rotational curve profile observed in several low surface brightness galaxies, reported in [13], where for computing the velocity due to the dark halo, $\vec{v}_{\rm halo}$, we are considering ${v_{\rm halo}}^2 = {v_{\rm observed}}^2 - {v_{\rm disk}}^2$, where $v_{\rm observed}$ is the magnitude of the observed velocity, and $v_{\rm disk}$ is the magnitude of the velocity computed due to the observed luminous distribution.

The procedure for a given galaxy, consists in determining the best value of the parameters β_0 and a that fits the observed rotational curve. Those values are reported in table I

Once $\beta^2(x)$ has been fixed for each galaxy, then solve the integral Eq. (8).

It is worth to notice that, for the (usual) case when $\beta_0 \ll 1$, the Eq. (8) can be readily solved:

$$\bar{p}_{\rm approx} = \frac{{\beta_0}^4}{6\,q}\,\frac{x^2+2\,a^2}{\left(x^2+a^2\right)^2}. \tag{18}$$

We use this approximate expression to evaluate the initial pressure $\bar{p}(0)$ in the full description, such as $\bar{p}(0) = \bar{p}_{\rm approx}(0) = \beta_0^4/3qa^2$. Hence the integration constant can be directly computed as

$$C = \bar{p}(0)a^{\frac{2\beta_0^2(\beta_0^2-1)}{\beta_0^2+1}} = \frac{\beta_0^4}{3q}a^{2\frac{\beta_0^4-2\beta_0^2-1)}{\beta_0^2+1}}\,. \tag{19}$$

	Galaxy	$\beta_0\,10^{-4}$	a	$\rho(0)\,10^{-4}$	$p(0)\,10^{-12}$	$\omega_{DM}\,10^{-9}$
1	UGC3137	3.68	4.09	162.28	733.98	45.42
2	UGC711	3.33	6.62	50.72	188.00	36.58
3	DD0189	1.95	2.71	103.08	130.69	12.86
4	NGC100	3.11	2.68	269.18	868.53	32.73
5	NGC2366	1.22	1.17	218.71	109.15	5.11
6	NGC3247	2.25	1.48	469.59	812.25	17.68
7	NGC5023	2.97	1.63	662.89	1953.05	30.09
8	NGC4455	2.58	6.21	34.57	76.73	21.96
9	UGC10310	2.82	4.03	97.57	258.25	26.59
10	UGC1230	2.83	3.89	106.02	283.71	26.91
11	UGC1281	2.33	3.17	108.54	197.01	18.35
12	UGC3371	2.85	8.14	24.53	66.45	26.49
13	UGC4173	1.84	12.46	4.39	5.01	10.88
14	UGC4325	5.11	6.27	133.26	1163.15	86.35
15	UGC5750	3.16	13.49	10.98	36.61	31.73
16	LSBF563	2.88	4.22	93.29	258.56	27.80

TABLE I: Fits for several galaxies, with $q = 5\,10^{-7}$, which corresponds to a system with distance in Kiloparsec units and with a halo mass at ten Kiloparsecs of 10^{10} Solar masses.

Our results for the mass density, the pressure and the resulting p vs ρ relation are shown in figure 1 for all LSB galaxies listed in Table I. In the bottom right panel of Fig. 1 we have included in shadow grey region the cosmological bounds obtained by Muller in [3] for a barotropic equation of state. It is interesting that the dark matter equation of state is fully determined by $\beta^2(x)$ and the election of $\bar{p}(0)$. If we use $\bar{p}(0)$ given by Eq. (19) and as the density profile Eq. (14), the resulting equation of state for dark matter are compatible with the cosmological bounds as can bee seen in the figure. Furthermore, to

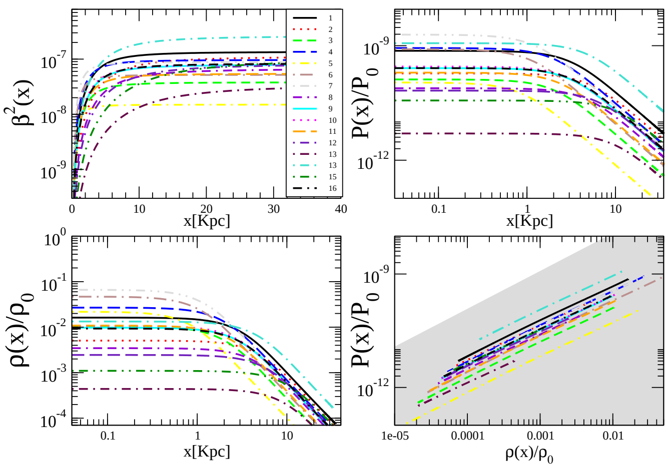

FIG. 1: Top left panel: Velocity profiles for each of the 16 LSB galaxies listed in table I. Top right panel: Pressure profiles. Bottom left panel: Density profiles. Bottom right panel: Pressure as a function of the density, gray area correspond to cosmological bounds on $P(\rho)$ [3].

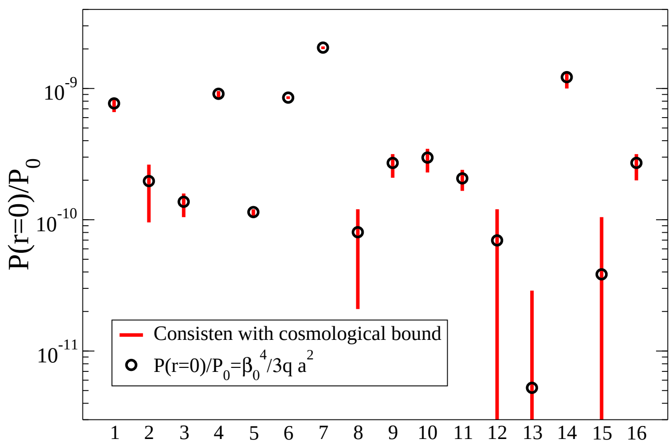

FIG. 2: Values for the central pressure for some typical dark matter dominated galaxies. Circles correspond to the election for the central pressure given by $\bar{p}(0) = \beta_0^4/3qa^2$. The red line shows the allowed region for the central pressure such as the resulting EOS satisfy both the positive energy condition and the cosmological bounds on barotropic equation of state found in [3].

first order there is a $\bar{p} = \omega_{DM}\bar{\rho}$ functional dependence of the pressure as a function of the density as can be seen in the same figure. The resulting values of ω_{DM} fitted from our the exact solutions for $\bar{p}$ are reported in last column of Table I.

We can ask ourselves how large can be the pressure at the center of the galaxy such as

1. $\rho(x) > 0$, i.e. fulfills positive energy condition and

2. $-1.5 \times 10^{-6}\bar{\rho}(x) < \bar{p}(x) < 1.13 \times 10^{-6}\bar{\rho}(x)$ i.e. fulfills cosmological bounds [3].

In Fig. 2 it is shown the pressure at the center of galaxy such that both conditions are considered.

Finally, once the solutions for the equation of state within general relativity were obtained, we can study the Newtonian limit in order to asses its accuracy. In this case, given a velocity profile $\beta^2(x)$ Eq. (14), one obtains directly the density and the mass function, and it is left, as before, to perform an integration for obtaining the pressure.

It is obtained the following expressions for the mass function, density and pressure analytically from Eqs. (11-13):

$$\bar{\rho} = \frac{{\beta_0}^2}{3\,q}\frac{\left(x^2 + 3\,a^2\right)}{\left(x^2 + a^2\right)^2}, \tag{20}$$

$$n = \frac{{\beta_0}^2}{q}\frac{x^3}{\left(x^2 + a^2\right)} \tag{21}$$

$$\bar{p} = \frac{{\beta_0}^4}{6\,q}\frac{\left(x^2 + 2\,a^2\right)}{\left(x^2 + a^2\right)^2}, \tag{22}$$

In the Newtonian case, it is possible to obtain an analytical expression for the equation of state for dark matter:

$$\bar{p} = \frac{\beta_0^2\bar{\rho}\left(\beta_0^2 + 6a^2q\rho + \beta_0\sqrt{\beta_0^2 + 24a^2q\rho}\right)}{\left(\beta_0 + \sqrt{\beta_0^2 + 24a^2q\rho}\right)^2} \tag{23}$$

In excellent agreement with the general relativistic results.

IV. CONCLUSIONS

We have modeled dark matter as a perfect fluid without assuming any specific form of the equation of state. We have shown that, in a spherically symmetric space, the rotation profile curve directly implies the pressure as a function of the density. By demanding dark matter to fulfill positive energy condition. a maximum pressure at the center of galaxies was obtained, namely:

$$p(0) < 1.22 \times 10^{-16}\left(\frac{v_0}{\mathrm{Km/sec}}\right)^4 \frac{1}{a^2}\frac{\mathrm{grms}}{\mathrm{cm\ sec}^2}\,, \tag{24}$$

where v_0 is the asymptotic circular velocity of the halo.

Furthermore, we have shown that for certain values of $\bar{p}(0)$, the resulting equation of state is consistent with cosmological bounds for a barotropic equation of state (see Fig. 2). Departure from the pressure-less hypothesis ($p \neq 0$) for the dark matter at the astrophysical and cosmological level may help us to understand the nature of the dark matter.

In this work, we shown that the observations make plausible that the dark fluid has a relation between the pressure and the density of the form $p = \omega_{DM}\,\rho$, where

ω_{DM} was obtained from a linear fit to the EOS obtained from each galaxy (See last column of Table I for numerical values). Actually, the density, ρ must be expressed in terms of the rest mass density, ρ_0, the one which satisfies the continuity equation, $(\rho_0\, u^\mu)_{;\mu} = 0$, and the internal energy of the fluid, ϵ, obtaining:

$$p = \omega_{DM}\ \rho_0\, c^2\ (1+\epsilon)\,. \tag{25}$$

If we deal with the dark fluid as if it were a perfect fluid, then we obtain an expression which allows us to relate the velocity parameter, ${\beta_0}^2$ (identified with ω_{DM}), in terms of well defined thermodynamical functions:

$$\frac{R\,T}{m_0\, c^2\ (1+\epsilon)} = {\beta_0}^2, \tag{26}$$

so that we can understand the dark matter halo as formed of a dark fluid with a temperature. The parameters of the observed rotational velocity profile are related to such temperature, so that there is not a different type of dark fluid for each halo, but the same dark fluid at different temperatures.

We will give more ground to this lucubration in a forthcoming work.

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